ON "VIRTUAL DISPLACEMENTS" OF MATERIAL SYSTEMS WITH LINEAR DIFFERENTIAL CONSTRAINTS OF THE SECOND ORDER

(O "VOZMOZHNYKH PEREMESHCHENIAKH MATERIAL'NYKH SISTEM S LINEINYMI DIFFERENTSIAL'NYMI SVIAZIAMI VTOROGO PORIADKA)

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V.I. KIRGETOV
(Moscow)

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The concept of "virtual displacement" of a system is, of course, a basic one in analytical mechanics. This is not just one of the concepts of analytical mechanics, but a concept on which the whole structure of analytical mechanics has been erected, a concept determining the character of analytical mechanics, the degree of its generality, the limits of its applications. Analytical mechanics extends only over those material systems for which the concept of "virtual displacements" has been established or, in other words, the "virtual displacements" have been defined.

In the development of analytical mechanics, several phases can be distinguished. Each of these phases had a corresponding definition of "virtual displacements" of a system. Thus, at present analytical mechanics contains several different definitions of "virtual displacements". Two remarks are justified in relation to all these definitions.

First, all definitions are linear. Their linearity lies in the fact that, according to these definitions, any linear combination of "virtual displacements" of a system is also a "virtual displacement" of this system.

Second, the "virtual displacements" introduced by these definitions do not depend on forces acting on the system. This property should be understood in the sense that any row $\delta x_1, \ldots, \delta x_{3n}$, being "virtual displacements" of a system under some forces, remain its "virtual displacements" for any other forces acting on this system.

The separate phases in the development of analytical mechanics are distinguished according to the degree of generality of the material systems being investigated. The transition from one phase to the following was accomplished by an extension of analytical emchanics to new systems. It was characterized by the introduction of a new definition of "virtual displacements" of a system, which made possible this extension of the scope of application of analytical mechanics. The newly introduced definition of "virtual displacements" of a system always contained the preceding one as a particular case. Thus, the transition from the preceding definition to the following one was a transition from a particular definition to a more general one. This process of successive generalization of the definition of "virtual displacements" possessed the remarkable peculiarity of not bringing analytical mechanics beyond the limits of one dynamical principle - the principle of Gauss.

The most general of all currently accepted definitions of "virtual displacements" is the known definition of Chetaev [1] for systems with non-linear differential constraints of the first order. In this paper, some extension of the definition of Chetaev is given. This extension is introduced within the principle of Gauss. On the basis of the linearity of the proposed definition and on the basis of the independence of the defined "virtual displacements" from the acting forces, a property of the proposed definition is proved. The significance of this property is that it may be used, in the specific sense, for the confirmation of the uniqueness of the proposed definition, and, in its framework, of all the previous definitions of "virtual displacements".

1. Consider a system of n material points. We denote by x_1, x_2, x_3, $m_1 = m_2 = m_3$; x_4, x_5, x_6, $m_4 = m_5 = m_6$; ... the Cartesian coordinates and masses of the first, scond, ... point respectively by X_1, X_2, X_3; X_4, X_5, X_6; ... the components of forces acting on the first, second ... point. All forces acting on the system (only active forces being considered) are assumed to be known functions of time, coordinates and velocities of the points of the system. We assume that at any given instant of time and at any given state of the system, the forces acting on the system may be changed in an arbitrary manner; here, as the state of the system at any instant of time we understand the positions and velocities of its points at this instant.

We assume the constraints of the system as being linear, and of the second order. It means, according to the terminology introduced by Delassus [2] , that the equations of constraints depend linearly on accelerations of the points of the system. Let the equations of the constraints of the system be

$$a_{\lambda 1} x_1'' + \ldots + a_{\lambda,\,3n} x_{3n}'' = a_\lambda \qquad (\lambda = 1, \ldots, m) \tag{1}$$

These equations are obviously assumed to be independent.

We assume that the constraints of the system do not depend on the forces acting on the system. This should be understood in the sense that the changes of the acting forces do not cause changes of the coefficients and of free terms in equations (1). In consequence of this assumption, the functions $a_{\lambda i}$ and a_λ depend only on time, coordinates and velocities of the points of the system.

Remark. It should be noted that this paper is not the first attempt to investigate material systems with constraints of the second order. Material systems of this type have been discussed, for instance, in the papers by Delassus [2], Przeborski [3], Valcovici [4].

2. According to the Gaussian principle, the set of accelerations of the points of the system corresponds to the minimum of the function

$$\sum_{i=1}^{3n} \frac{m_i}{2}\left(x_i'' - \frac{X_i}{m_i}\right)^2$$

with the condition of satisfaction of the relations

$$\sum_{i=1}^{3n} a_{\lambda i} x_i'' = a_\lambda \qquad (\lambda = 1, \ldots, m)$$

This implies that the actual accelerations of the points of the material system should satisfy the equations

$$m_i x_i'' - X_i + \sum_{\lambda=1}^{m} \sigma_\lambda a_{\lambda i} = 0, \qquad \sum_{i=1}^{3n} a_{\lambda i} x_i'' = a_\lambda \qquad \begin{pmatrix} i = 1, \ldots, 3n \\ \lambda = 1, \ldots, m \end{pmatrix} \tag{2}$$

where the coefficients σ_λ represent the undetermined Lagrangian multipliers,

Introducing the notations

$$b_{\lambda j} = \frac{a_{\lambda j}}{m_j}, \qquad b_\lambda = a_\lambda - \sum_{i=1}^{3n} a_{\lambda i} \frac{X_i}{m_i} \tag{3}$$

we write equations (2) in the form

$$\begin{aligned} m_i x_i'' - X_i + \sum_{\lambda=1}^{m} \sigma_\lambda b_{\lambda, i} m_i &= 0 \\ \sum_{i=1}^{3n} b_{\lambda i}(m_i x_i'' - X_i) &= b_\lambda \end{aligned} \qquad \begin{pmatrix} i = 1, \ldots, 3n \\ \lambda = 1, \ldots, m \end{pmatrix}$$

From here the equations for the undetermined multipliers follow immediately

$$\sum_{\lambda=1}^{m} \sigma_\lambda \sum_{i=1}^{3n} b_{\mu i} b_{\lambda i} m_i = -b_\mu \qquad (\mu = 1, \ldots, m) \tag{4}$$

Since the equations of constraint are independent, the determinant of System (4) is different from zero. Consequently, this set of equations allows us to find the undetermined multipliers σ_λ.

Equations (2), in which multipliers are found from equations (4), allow us to determine the accelerations of the system at any instant of time, if only the state of the system and the forces acting on it (no matter what they are) are given at this instant.

3. We assume as "virtual displacements" of the system all possible rows $\delta x_1, \ldots, \delta x_{3n}$ whose components satisfy the relations

$$\sum_{i=1}^{3n} a_{\lambda i}\delta x_i = 0 \qquad (\lambda = 1, \ldots, m) \tag{5}$$

where coefficients $a_{\lambda i}$ are the coefficients of equations (1).

We will denote this definition of "virtual displacements" of the system by letter A. We will show that the definition A has all the properties, pointed out in the introduction, which are common to all definitions of "virtual displacements" of a system, accepted in analytical mechanics.

It is obviously linear.

Then, the "virtual displacements" of a system, introduced by this definition, do not depend on the forces acting on the system.

In fact, the coefficients $a_{\lambda i}$, by assumption, do not depend on the forces acting on the system. Thus any row $\delta x_1, \ldots, \delta x_{3n}$, satisfying relations (5) for some forces and being the "virtual displacements" for these forces, will satisfy them and will be the "virtual displacements" of the system also for any other forces acting on the material system considered.

Finally, the definition A is such that the d'Alembert-Lagrange principle yields, within this definition, the same equations as does the Gaussian principle.

To prove this, we will show that the d'Alembert-Lagrange principle, within the definition A, yields precisely equations (2). The derivation of the equations of motion from the d'Alembert-Lagrange principle will be performed in the usual way. We multiply equations (5) by the undetermined multipliers σ_λ and add them to the fundamental equations of mechanics. We obtain

$$\sum_{i=1}^{3n} (m_i x_i'' - X_i)\,\delta x_i + \sum_{\lambda=1}^{m} \sigma_\lambda \sum_{i=1}^{3n} a_{\lambda i}\delta x_i = 0$$

which may be rewritten in the form

$$\sum_{i=1}^{3n} (m_i x_i'' - X_i + \sum_{\lambda=1}^{m} \sigma_\lambda a_{\lambda i})\, \delta x_i = 0 \tag{6}$$

Hence, with a proper selection of the multipliers σ_λ (the multipliers σ_λ should be such that, in relations (6), the expressions within parenthesis vanish) there follow immediately the equations

$$m_i x_i'' - X_i + \sum_{\lambda=1}^{3n} \sigma_\lambda a_{\lambda i} = 0 \qquad (i = 1, \ldots, 3n) \tag{7}$$

Equations (7), to which the equations of constraint are to be added, coincide completely with equations (7), derived from the Gaussian principle. This completes the proof.

4. Two definitions of "virtual displacements" of a system will be called equivalent, if the sets of "virtual displacements" determined by these definitions are identical.

The definitions of "virtual displacements" given by the relations

$$\sum_{i=1}^{3n} a_{\lambda i} \delta x_i = 0 \qquad (\lambda = 1, \ldots, m) \tag{8}$$

$$\sum_{i=1}^{3n} b_{\lambda i} \delta x_i = 0 \qquad (\lambda = 1, \ldots, m) \tag{9}$$

respectively, where

$$b_{\lambda i} = \sum_{\mu=1}^{m} c_{\lambda\mu} a_{\mu i}$$

are obviously equivalent, (the determinant $|c_{\lambda\mu}|$ being different from zero).

On the other hand, if the definitions of "virtual displacements", given by relations (8) and (9), respectively, are equivalent, then all the relations (9) may be represented as linear combinations of relations (8).

In fact, relations (8) and (9) may be considered as two sets of linear algebraic equations. These two sets have, according to the assumption, identical solutions and consequently, as it is known from algebra, the equations of one set, for instance equations (9), will be linear combinations of the equations of another set - the set (8).

We will prove the following proposition.

If "virtual displacements" do not depend on the forces acting on the system, and the d'Alembert-Lagrange principle yields the same equations of motion as does the Gaussian principle, then there does not exist any linear definition of "virtual displacements" which would not be equivalent to the definition *A*.

Let *B* be an arbitrary linear definition of "virtual displacements" of the system, satisfying the assumption of the above proposition.

The argument will be presented for any given instant of time and the corresponding state of the system.

Let the row

$$\delta\alpha_1, \ldots, \delta\alpha_{3n} \tag{10}$$

be the "virtual displacements" at some forces, according to the definition *B*.

Thus, for these forces, the relation

$$\sum_{i=1}^{3n} (m_i x_i'' - X_i)\delta\alpha_i = 0 \tag{11}$$

(x_i'' here are the actual accelerations of the system) is satisfied, and consequently, the relation

$$\sum_{i=1}^{3n} \delta\alpha_i \sum_{\lambda=1}^{m} \sigma_\lambda a_{\lambda i} = 0$$

is also satisfied, obtained from (11) by eliminating by means of equations (2) the quantities $m_i x_i'' - X_i$.

The last relation may be rewritten in the form

$$\sum_{\lambda=1}^{m} \sigma_\lambda \sum_{i=1}^{3n} a_{\lambda i}\delta\alpha_i = 0 \tag{12}$$

If the initially given forces are replaced by other ones, the row (10) does not cease to be the "virtual displacements" of the system, and therefore, relation (12) does not cease to be valid for this transition to new forces. It implies that it is valid for any change of forces acting on the system. For these changes of forces, the sums

$$a_{\lambda 1}\delta\alpha_1 + \ldots + a_{\lambda, 3n}\delta\alpha_{3n}$$

being independent of the forces, remain invariant while the quantities σ_λ change arbitrarily. In fact, the system of equations (4) may be written as

$$\sum_{i=1}^{3n} X_i b_{\mu i} = a_\mu + \sum_{\lambda=1}^{m} \sigma_\lambda \sum_{i=1}^{3n} b_{\mu i} b_{\lambda i} m_i \qquad (\mu = 1, \ldots, m)$$

The last set of equations determines the forces related to any choice of multipliers σ_λ.

From what was said above, it follows that the following equations should be satisfied

$$\sum_{i=1}^{3n} a_{\lambda i}\delta\alpha_i = 0 \qquad (\lambda = 1, \ldots, m)$$

This means that any "virtual displacements" of the system determined by definition B will be also the "virtual displacements" in the sense of definition A.

On the other hand, the number of equations of motion obtained from the d'Alembert-Langrange principle, for any definition of "virtual displacements", is equal to the number of linearly independent "virtual displacements". Thus, the number of linearly independent virtual displacements among all the "virtual displacements" given by definition B, is equal to $3n - m$, i.e. it is equal to the number of linearly independent "virtual displacements" given by definition A. From this and from the theorem proved, it follows that the sets of "virtual displacements" of a system given by the definition A and B should be identical, and consequently, definition B should be equivalent to definition A. This completes the proof.

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